Exercise 2

1. If a is a positive integer, then $a^2 + a^4 \equiv 0(mod 5)$ if

   (a) $a \equiv 2(mod 5)$
   (b) $a \equiv 3(mod 5)$
   (c) 5 divides a
   (d) None of the above

   Ans. (a),(b) and (c). Check by substitution

2. If n is a positive integer, such that $n^2 \equiv x \pmod 8$. Which of the following is/are possible value(s) of x.

   (a) 1
   (b) 2
   (c) 3
   (d) 5

   Ans: (a)

   Let $n = 4k + r$, then $n^2 = 16k^2 + 8kr + r^2 \equiv r^2 \pmod 8$. Hence, check for $r \in \{0, 1, 2, 3\}$

3. If p is a prime, then p can be congruent to which of the following modulo 12.

   (a) 2
   (b) 5
   (c) 7
   (d) 9

   Ans: (a), (b) and (c). $p = 2$, $p = 5$, $p = 7$ are examples for the first three options respectively. (d) cannot be the answer as 3 divides all numbers of the form $12k + 9$.

4. If a and b are real number then, the statement $\frac{a+b}{2} \geq \sqrt{ab}$

   (a) is always true
   (b) is true only when a and b are positive
   (c) is true only when $a = b$
   (d) is always false

   Ans. (b)

   $(a - b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + 2ab + b^2 \geq 4ab \Rightarrow \frac{(a+b)^2}{4} \geq ab$ If a and b are positive taking square-root on both sides does not change the inequality $\frac{a+b}{2} \geq \sqrt{ab}$

5. If x, y are two real numbers then which of the following is/are true

   (a) $x^2 + y^2 \geq 2xy$
   (b) $x^2 + y^2 \geq 2xy$ or $x^2 + y^2 \leq xy$
   (c) $x^2 + y^2 \geq 2xy$ and $x^2 + y^2 \geq 2x + 2y - 2$
   (d) None of the above

   Ans. (a), (b) and (c)

   $(x - y)^2 \geq 0 \implies x^2 + y^2 - 2xy \geq 0$
   $(x - 1)^2 + (y - 1)^2 \geq 0 \implies x^2 + y^2 - 2x - 2y + 1 + 1 \geq 0$

6. If a and b are two primes ($a \neq b$ and $a, b \geq 2$) then which of the following is/are true.

   (a) $a^2 + b^2 \geq 13$
   (b) $a^2 + b^2 \geq 34$ or ab is even
   (c) $(a + b)^2 \geq 4ab$
   (d) None of the above

   Ans: (a), (b) and (c).

   The two smallest primes satisfying the condition are 2 and 3, hence $a^2 + b^2$ is at least 13.
   If ab is odd, the two smallest primes satisfying the condition are 3 and 5, hence $a^2 + b^2$ is at least 34.
   $(a + b)^2 \geq 4ab$ for all a and b

7. If n is a positive integer, such that $n^2 \equiv x(mod 12)$. Which of the following is/are possible value(s) of x.

   (a) 1
   (b) 2
   (c) 21
   (d) 3

   Ans: (a) and (c). Let $n = 6k + r$. This problem is then similar to problem 2.

8. Let a, b, c be the 3 sides of a right angled triangle with c as the hypotenuse. If a, b, c are natural numbers then which of the following is/are true:

   (a) $a^2 + b^2 = c^2$
   (b) Area of this right-angled triangle is always divisible by 3
   (c) Area of this right-angled triangle is always divisible by 5
   (d) $a^2 + b^2 \leq c^2$

Ans: (a), (b) and (d).

Clearly (a) and (d) are correct by Pythagoras Theorem. (c) is not correct since 3, 4, 5 form a right angle triangle with area 6.

Every perfect square is of the form $3k$ or $3k+1$ (Proof similar to Problem 2). Hence, if c^2 is of the form $3k$, then a and b are both divisible by three (Since $a^2 + b^2 = c^2$), hence the area is divisible by three. Else, one of a^2 or b^2 is of the form $3k + 1$ and the other of form $3k$. Hence, the area is divisible by three. (remember area = ab/2)